Ensemble-averaged conductance fluctuations in Anderson-localized systems

M. Hilke*
Department of Physics, McGill University, Montréal, Québec, Canada H3A 2T8
and Institute for Theoretical Physics, Technische Universität Berlin, 10623 Berlin, Germany
(Received 12 June 2008; published 24 July 2008)

We demonstrate the presence of energy dependent fluctuations in the localization length, which depend on the disorder distribution. These fluctuations lead to ensemble-averaged conductance fluctuations (EACFs) and are enhanced by large disorder. For the binary distribution the fluctuations are strongly enhanced in comparison to the Gaussian and uniform distributions. These results have important implications on ensemble-averaged quantities, such as the transmission through quantum wires, where fluctuations can subsist to very high temperatures. For the nonfluctuating part of the localization length in one dimension we obtained an improved analytical expression valid for all disorder strengths by averaging the probability density.

DOI: 10.1103/PhysRevB.78.012204 PACS number(s): 73.63.−b, 73.20.Fz, 73.21.Hb, 74.40.+k

In quantum coherent conductors, disorder induces dramatic effects on the conductance. For instance, if the localization length L_c, which is the decay length of the wave function, is exceeded by the system size L, the conductance vanishes exponentially (Anderson localization).[1] On the other hand, if the system size is smaller than L_c, then the conductance exhibits fluctuations in energy which take on universal amplitudes and are referred to as universal conductance fluctuations (UCFs).[2]

In many quantum wires, such as carbon nanotubes, the conductance can be measured as a function of the gate voltage, effectively changing the Fermi energy of the carriers. As a function of energy, the conductance shows strong fluctuations associated with disorder.[3,4] The amplitude of these fluctuations decays with temperature due to the decrease of the quantum coherence length and the resulting effective ensemble average.[2] However, in some experimental systems, the conductance fluctuations remain substantial even at high temperatures (equivalent to ensemble averaging).[5]

In order to better understand possible fluctuations even after ensemble averaging, we studied in detail the energy dependence of the ensemble-averaged quantity L_c. Quite strikingly, we observe fluctuations of L_c as a function of energy, which depend on the distribution of the disorder potential. This is illustrated in Figs. 1 and 2, where we show the relative variation of the inverse localization length (Lyapunov exponent λ) as a function of energy for different disorder distributions (Gaussian, uniform, and binary). For the calculations we used the one-dimensional Anderson model, described in more detail below.

The variations (or fluctuations) of the Lyapunov exponent are quite substantial for the binary distribution and reach up to 80%. The fluctuations are shown with respect to the Lyapunov exponent (λ_ρ) obtained from the average probability density, valid for all disorder strengths and derived below. The symmetry of the fluctuations depends on the symmetry of the distribution. Indeed, for $\alpha=1/2$ the distribution is symmetric in energy, which leads to symmetric fluctuations as seen in Fig. 1 in stark contrast to the skewed case $\alpha \neq 1/2$, where the distribution is asymmetric.

In Fig. 2 we show the relative deviations of the Lyapunov exponent for the Gaussian and uniform distributions, where W is the width of the uniform distribution. Here the fluctuations are strongly suppressed and the deviations are of the order of 10% and are very smooth when compared to the binary distributions for the same standard deviation range. However, the observed strong fluctuations for the binary distributions lead us to make a comparison with mesoscopic conductance fluctuations.

In general, conductance fluctuations are expressed as $\langle \delta G^2 \rangle = \langle (G - \langle G \rangle)^2 \rangle$, where $\langle \cdot \rangle$ denotes the ensemble average, i.e., the average over a given disorder distribution. The standard universal conductance fluctuations result gives $\sqrt{\langle \delta G^2 \rangle} \simeq 0.73 e^2/h$ (Ref. 2) for a quasi-one-dimensional system. In the localized regime these fluctuations are suppressed. In order to characterize the fluctuations in the Lyapunov exponent seen in Figs. 1 and 2, we define

$$\delta\lambda^2 = \frac{1}{E_{\max} - E_{\min}} \int_{E_{\min}}^{E_{\max}} [\lambda(E) - \lambda_\rho(E)]^2 dE, \qquad (1)$$

where λ_ρ is the Lyapunov exponent obtained from the average probability density with the same standard deviation.

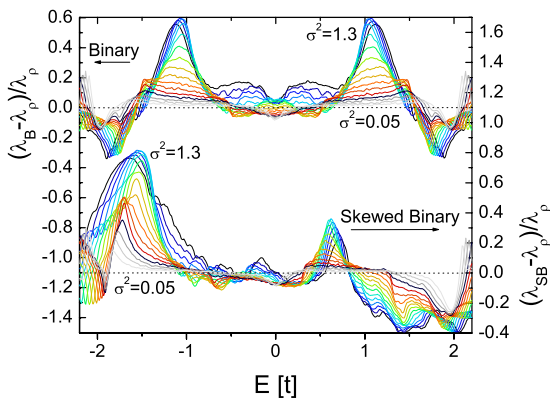

FIG. 1. (Color online) The relative deviation of the Lyapunov exponent for the binary and skewed binary distribution. The skewed binary distribution function of average zero is given by $P(V)=(1-\alpha)\delta(V-\alpha W)+\alpha\delta[V+(1-\alpha)W]$, where we have $\alpha=1/2$ for the binary case (λ_B) and we chose $\alpha=1/3$ for the skewed case (λ_{SB}). The variance σ^2 is varied between 0.05 and 1.3 and depends on W. λ_ρ is the Lyapunov exponent obtained from the average probability density.

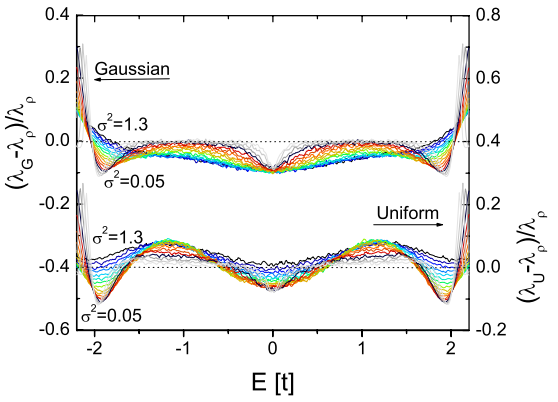

FIG. 2. (Color online) The relative deviation of the Lyapunov exponent for the Gaussian distribution (λ_G) of zero average and of standard deviation σ and the uniform distribution (λ_U with $-W/2 < V < W/2$ and $\sigma^2 = W^2/12$). σ^2 is varied between 0.05 and 1.3.

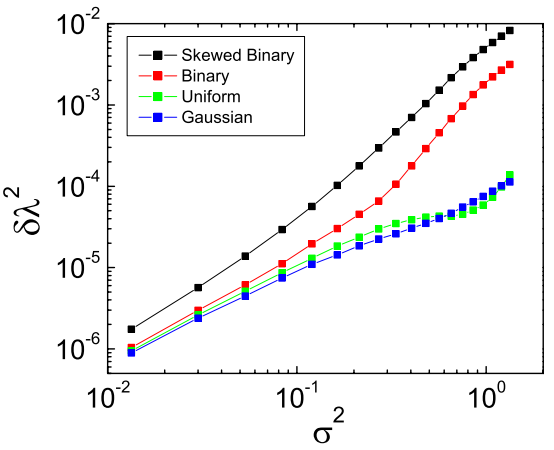

FIG. 3. (Color online) $\delta\lambda^2$ is shown for different distributions as a function of the disorder strength, which is characterized by the variance σ^2. For the skewed binary distribution we used $\alpha = 1/3$ (the same as in Fig. 1). These fluctuations represent ensemble-averaged conductance fluctuations.

The bandwidth ($E_{\max} - E_{\min}$) is fixed to the nondisordered one.

For a strongly localized system of size L, the typical conductance is given by $G_{\text{typ}} \sim \exp^{-\lambda L/2}$.[6] Hence, the fluctuations in λ lead to fluctuations in the conductance, which we coin ensemble-averaged conductance fluctuations (EACFs). We studied EACFs for different disorder strengths. The results are shown in Fig. 3 for different distributions, which all show an increase of EACFs with the disorder strength. The increase is found to be more pronounced for the binary distributions.

We now describe the derivation of the Lyapunov exponent (λ_ρ) for the average probability density, which is used in the calculations of the relative fluctuations shown in Fig. 3. This is an analytical approach in order to obtain an expression for $\lambda(E)$ valid for all disorder strengths.

The one-dimensional Anderson model[1] is a tight-binding equation with random on-site potentials V_n and given by

$$t\Psi_{n+1} + t\Psi_{n-1} = (E - V_n)\Psi_n. \tag{2}$$

Here t is the hopping term, which we set to 1. In this model, an alloy of two elements would be described with V_n taken from a binary distribution such as in Fig. 1. In contrast, if the disorder is due to the surface roughness of the substrate, such as in the case of a carbon nanotube on a silicon oxide substrate, V_n would be given by a more continuous distribution, such as the Gaussian or uniform one.

In order to obtain $\lambda(E)$ the main idea is to obtain an iterative equation for the probability density and then to average it. Hence, assuming real potentials V_n it is possible to rewrite Eq. (2) as

$$\rho_{n+1} = \left[(E - V_n)^2 - \frac{E - V_n}{\{E - V_{n-1}\}}\right]\rho_n + [1 - (E - V_n)(E - V_{n-1})]\rho_{n-1} + \left[\frac{E - V_n}{E - V_{n-1}}\right]\rho_{n-2} \tag{3}$$

$$= [(E - V_n)^2 - 1]\rho_n + \left[1 - 2(E - V_n)(E - V_{n-1}) + (E - V_{n-1})^2 + \frac{V_n - V_{n-1}}{\{E - V_{n-2}\}}\right]\rho_{n-1}$$

$$+ [(E - V_n)(E - V_{n-2}) + 1 - (E - V_{n-1})(E - V_{n-2})]\rho_{n-2} + \left[\frac{V_{n-1} - V_n}{E - V_{n-2}}\right]\rho_{n-3}, \tag{4}$$

where $\rho_n = \psi_n \psi_n^*$ is the probability density. Equation (4) is obtained by using an additional iteration and will be important when considering the average. Interestingly, this exact (before averaging) iterative expression for the probability density depends explicitly on neighboring potentials, which illustrates the importance of the assumption of an uncorrelated disorder potential when taking the disorder average. Indeed, the presence of local correlations can lead to an infinite localization length in one- and two-dimensional disordered systems.[7] In our method of considering the probability density, ρ_n instead of ψ_n, phase correlations are not averaged out when the disorder average is performed.

The average of Eq. (4) can be taken trivially by assuming uncorrelated disorder with $\langle V_n V_m \rangle = \sigma^2 \delta_{n,m}$ and noting that $\langle \rho_i V_j \rangle = \langle \rho_i \rangle \langle V_j \rangle$ for $i \geq j$, since ρ_n does not depend on V_n when ρ_n is obtained iteratively using initial conditions for

$n=0,1,2,3$. The term in curly brackets of ρ_n in Eq. (3) is the only product where the averages cannot be factorized, since $\langle \rho_n/(E-V_{n-1})\rangle \neq \langle \rho_n\rangle/\langle (E-V_{n-1})\rangle$, because ρ_n depends on V_{n-1}. This is the reason we had to iterate this equation one more time in order to obtain Eq. (4), where the coefficient in front of the curly bracket term now averages to zero.

For a potential of average zero we thus obtain the following iterative solution for the average probability density:

$$\langle \rho_{n+1}\rangle = (E^2+\sigma^2-1)\langle \rho_n\rangle + (1-E^2+\sigma^2)\langle \rho_{n-1}\rangle + \langle \rho_{n-2}\rangle. \tag{5}$$

The leading dependence of $\langle \rho_n\rangle$ can now be extracted by evaluating the eigenvalues $\{\xi_1,\xi_2,\xi_3\}$ of the characteristic transfer matrix determined by Eq. (5). The corresponding Lyapunov exponents are

$$\lambda_i = \frac{1}{2}|\log(|\xi_i|)|, \tag{6}$$

where the factor 1/2 comes from the definition of the Lyapunov exponent in terms of the probability density instead of the wave function ($\rho_i=|\psi_i|^2$). For $\sigma=0$ all three Lyapunov exponents are zero, which is characteristic of the plane-wave solution for an energy within the band $-2\leq E\leq 2$. Defining $\lambda_1\leq\lambda_2\leq\lambda_3$ we have for $\sigma>0$ that $\lambda_1>0$, which implies that all states are localized. The relevance of each Lyapunov exponent depends on the physical quantity. For instance, the largest Lyapunov exponent (in our case λ_3) determines the dependence of the average resistance ($\langle R\rangle \sim e^{2\lambda_3 L}$) as discussed by Pendry.[6] Pendry[6] and Erdos and Herndon[8] used symmetry properties of generalized transfer matrices to obtain a similar structure resulting in three Lyapunov exponents. Here we are interested in the leading behavior of the wave function (and not the resistance) and therefore use the following scheme to extract λ_ρ.

For energies close to $E=0$ we define λ_ρ as the average of the two smallest Lyapunov exponents, i.e., $\lambda_\rho\equiv(\lambda_1+\lambda_2)/2$. Away from the band center, corresponding to $|E|\gtrsim\sigma^2/4$, we use $\lambda_\rho\equiv\lambda_1=\lambda_2$ and for $|E|\gtrsim 2+3\sigma^{2/3}/4$ ($\sigma\ll 1$) we have $\lambda_1\neq\lambda_2$ and define $\lambda_\rho\equiv\lambda_2$ in this range. With these definitions we obtain $\lambda_\rho\simeq(1/2)\sigma^2/(4-E^2)=\lambda_T(E)$ in the limit of small disorder ($\sigma\ll 1$), where λ_T corresponds to the standard result by Thouless.[9] Thouless obtained this result by averaging the Green's function to second order. The above definitions can be understood in terms of the positivity requirement of ρ_n, where we have to exclude λ_1 as leading dependence in some of these energy intervals. The dependence on energy of all three Lyapunov exponents are shown in the bottom panel of Fig. 4.

The band center ($E=0$) constitutes a special case, where the correction for the small disorder expansion[10,11] is given by $\lambda_{\text{K-W}}(E=0)\simeq 0.91\lambda_T(0)$, where $\lambda_{\text{K-W}}$ is the correction due to the Kappus-Wegner anomaly. This small disorder anomaly manifests itself in our probability density approach through expressions (3) and (4), which are ill defined when taking the average at $E=0$ and lead to the singular result $\lambda_1(E=0)=0$. Other small disorder anomalies occur at the band edges, where corrections to the Thouless result were obtained recently.[12] These anomalies at the band center and at the band edges can be easily identified in Fig. 2 for the Gaussian distribution, where they appear as troughs for small disorder. Most previous analytical approaches are based on small disorder expansions and cannot be applied to high disorder. This is in contrast to our probability density approach, where we did not assume small disorder, and we indeed obtain the correct large disorder limit $\lambda_\rho\equiv\lambda_2\simeq\lambda_3\simeq\log(\sigma)$ with a Lyapunov exponent independent of E.[13]

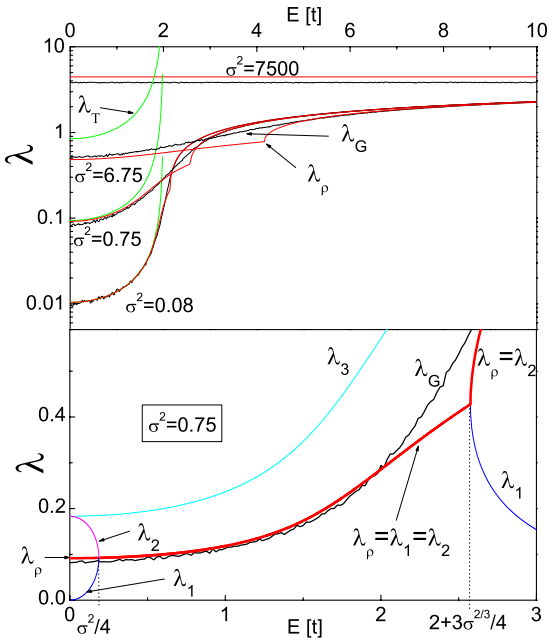

FIG. 4. (Color online) Upper panel: Lyapunov exponents (λ_G, λ_ρ, and λ_T) as a function of energy for different disorder strengths. The labeled curves correspond to $\sigma^2=6.75$. Lower panel: λ_1, λ_2, and λ_3 are shown together with λ_G and λ_ρ for $\sigma^2=0.75$.

Numerically, the Lyapunov exponent was obtained by evaluating the eigenvalues of the product of transfer matrices obtained from Eq. (2),

$$\{\xi_i^N\} = \text{eig}\left[\prod_{n=1}^{N}\begin{pmatrix} E-V_n & -1 \\ 1 & 0\end{pmatrix}\right] \tag{7}$$

and then taking the self-averaging limit where we used $N\gg 10^6$:

$$\lambda = \lim_{N\to\infty}\frac{1}{N}\log(\max\{|\xi_i^N|\}) = -\lim_{N\to\infty}\frac{1}{N}\log(\min\{|\xi_i^N|\}). \tag{8}$$

The quality of our expression for λ_ρ can be seen in Fig. 2, where we show the relative deviation to the numerically obtained Lyapunov exponents $\lambda_{G,U}$. The deviations are of the same order as the dependence on the distribution function (Gaussian versus uniform). This shows that any further improvement of the analytical expression for $\lambda(E)$ needs to depend on the distribution function explicitly (for example, by including higher moments). For the special case of the Cauchy distribution, where the second moment does not exist, an exact analytical expression for the Lyapunov exponent can be found.[14]

We also compared our expression λ_ρ in Fig. 4 for a large range of disorder strengths and find that the fit is less than a factor of 2 off for all energies and disorder strengths, in contrast to the Thouless expression (λ_T) also shown, which deviates substantially at large disorder.

The ensemble-averaged fluctuations, which is the main result presented here, were obtained for one of the simplest models showing Anderson localization, namely the random one-dimensional tight-binding equation, which describes single-channel quantum wires. However, Anderson localization is very general and can be observed, among others, in the propagation of light in disordered media, in phonon and plasmon modes, in quantum chaotic systems,[15] in Bose-Einstein condensates,[16] and even in neutron propagation.[17] We therefore expect that similar EACFs also exist in these systems, since the equations describing these are very similar to the tight-binding model. For higher dimensions, only very few analytical expressions exist, including a recent result valid for small disorder, which has been derived for quasi-one-dimensional systems[18] and that can be expressed as a convolution of the one-dimensional case. We therefore expect that at higher disorder strengths, distribution dependent fluctuations will also appear in quasi-one-dimensional systems, which are relevant for many experiments on quantum wires. More striking examples might be systems composed of alloys, since the potentials of a two-component alloy would be described by a binary disorder distribution, which shows the strongest EACFs. Indeed, in InGaAs alloys, conductance fluctuations in a quasi-one-dimensional geometry were observed at high temperatures.[5] However, the study of EACFs in higher dimensions is beyond the scope of this work.

*hilke@physics.mcgill.ca

[1] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[2] P. A. Lee and A. D. Stone, Phys. Rev. Lett. **55**, 1622 (1985); A. B. Altshuler, JETP Lett. **41**, 648 (1985).
[3] B. Stojetz, Ch. Hagen, Ch. Hendlmeier, E. Ljubovic, L. Forro, and Ch. Strunk, New J. Phys. **6**, 27 (2004).
[4] H. T. Man and A. F. Morpurgo, Phys. Rev. Lett. **95**, 026801 (2005).
[5] B. Hackens, F. Delfosse, S. Faniel, C. Gustin, H. Boutry, X. Wallart, S. Bollaert, A. Cappy, and V. Bayot, Phys. Rev. B, **66**, 241305(R) (2002).
[6] J. B. Pendry, Adv. Phys. **43**, 461 (1994).
[7] J. C. Flores, J. Phys.: Condens. Matter **1**, 8471 (1989); D. H. Dunlap, H.-L. Wu, and P. W. Phillips, Phys. Rev. Lett. **65**, 88 (1990); M. Hilke, J. Phys. A **27**, 4773 (1994); Phys. Rev. Lett. **91**, 226403 (2003).
[8] P. Erdos and R. C. Herndon, Adv. Phys. **31**, 65 (1982); B. U. Felderhof, J. Stat. Phys. **43**, 267 (1986).
[9] D. Thouless, in *Ill-Condensed Matter*, edited by R. Balian, R. Maynard, and G. Toulouse (North-Holland, Amsterdam, 1979).
[10] M. Kappus and Z. Wegner, Z. Phys. B: Condens. Matter **45**, 15 (1981).
[11] B. Derrida and E. Gardner, J. Phys. (France) **45**, 1283 (1984).
[12] L. Tessieri and F. M. Izrailev, Phys. Rev. E **62**, 3090 (2000); H. Schomerus and M. Titov, *ibid.* **66**, 066207 (2002); H. Schomerus and M. Titov, Phys. Rev. B **67**, 100201(R) (2003).
[13] D. Thouless, J. Phys. C **5**, 77 (1972).
[14] K. Ishii, Suppl. Prog. Theor. Phys. **53**, 77 (1973).
[15] *Wave Scattering in Complex Media: From Theory to Applications*, edited by B. A. van Tiggelen and S. E. Skipetrov (Kluwer, Dordrecht, 2003).
[16] B. Damski, J. Zakrzewski, L. Santos, P. Zoller, and M. Lewenstein, Phys. Rev. Lett. **91**, 080403 (2003).
[17] Yu. N. Pokotilovski, Eur. Phys. J. B **3**, 105 (1998); Ch. Stellmach, H. Abelea, A. Bouchera, D. Dubbersb, U. Schmidta, and P. Geltenbort, Nucl. Instrum. Methods Phys. Res. A **440**, 744 (2000).
[18] H. Schulz-Baldes, Geom. Funct. Anal. **14**, 1089 (2004); R. Roemer and H. Schulz-Baldes, Europhys. Lett. **68**, 247 (2004).